H2 Mathematics (9758)

Sequences and Series — EXAM Notes

A-Level 2027 Syllabus

Command Words & What They Require

Examiners use precise command words. Misinterpreting these is one of the most common sources of lost marks.

Command Word Find

Determine — show working, but no justification of method required. A formula can be quoted and applied. *Example: "Find the sum of the first 20 terms of the arithmetic series."*

Command Word Show that

You must present a clear chain of reasoning leading to the given result. All algebraic steps must be shown — quoting a final answer without intermediate working is insufficient.

Command Word Prove

A formal justification is required. For series: proof by induction (for recurrence relations) or algebraic manipulation showing convergence conditions.

Command Word Express

Write in a specified form — typically a simplified closed-form expression. *Example: "Express S_n in terms of n."*

Command Word Determine

Find a value or condition with reasoning. Often used for convergence tests or finding parameters in a series.

Command Word State

No working required — just write the answer. *Example: "State the range of values of x for which the geometric series converges."*

Command Word Hence

The next part of the question **must** use the result from the previous part. Failing to reference the earlier result loses method marks. *Example: "Hence find the sum to infinity of the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$."*

Command Word Find the sum

Evaluate a series using either the appropriate formula or sigma notation. Always state which formula you are using.

Mark Allocation Patterns

Typical 9758 Paper 1 Distribution

Sequences and Series typically appears in Paper 1 as either:

- **One 6–8 mark question** (method of differences or sigma notation with algebraic manipulation)
- **One 4–5 mark part within a longer question** (financial maths or recurrence relations)
- **Total: 8–12 marks per paper**, roughly 8–12% of Paper 1

How Marks Are Distributed

Question Type	Mark Breakdown Pattern
Arithmetic / Geometric series (find a, d or r, then S_n)	2M: forming equations from given conditions 2M: solving simultaneous equations 1M: correct formula substitution 1A: final answer
Sum to infinity	1M: stating convergence condition $\lvert r\rvert < 1$ 1M: substituting into $S_\infty = \frac{a}{1-r}$ 1A: final simplified answer
Method of differences	2M: splitting into partial fractions 2M: writing out the telescoping sum 1M: cancelling terms 1A: final expression for S_n
Recurrence relation	1M: finding first few terms 1M: conjecturing formula 1M: proving by induction (base case + inductive step)
Sigma notation evaluation	1M: identifying series type 1M: applying correct formula 1M: evaluating limits correctly 1A: final simplified answer

Exam Tip Read the mark scheme

A question worth 5 marks that asks you to "Find the sum of the series" expects you to show the formula, substitution, simplification, and final answer. **Dont skip steps — you lose marks.**

Question Templates with Worked Solutions

Arithmetic Series Problem

Example Arithmetic Series — Standard

The third term of an arithmetic progression is 10 and the seventh term is 22. Find

(a) the first term and common difference,

(b) the sum of the first 15 terms.

Solution.

(a) For an AP: $u_n = a + (n-1)d$.

$$u_3 = a + 2d = 10 \quad (1)$$
$$u_7 = a + 6d = 22 \quad (2)$$

Subtract (1) from (2): $4d = 12 \implies d = 3$.
Substitute back: $a + 2(3) = 10 \implies a = 4$.
Therefore $a = 4$, $d = 3$.

(b) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$.

$$S_{15} = \frac{15}{2}\left[2(4) + (14)(3)\right] = \frac{15}{2}\left[8 + 42\right] = \frac{15}{2} \times 50 = 375.$$

Exam Tip Set up two equations

When given two terms of an AP, you always get two linear equations in a and d. Solve by elimination. Never guess — the question is designed to be solved systematically.

Geometric Series Problem

Example Geometric Series — Sum to Infinity

A geometric progression has first term 8 and sum to infinity 12. Find

(a) the common ratio,

(b) the fifth term,

(c) the least number of terms for which the sum exceeds 11.5.

Solution.

(a) $S_\infty = \dfrac{a}{1-r} = 12$ with $a = 8$.

$$\frac{8}{1-r} = 12 \implies 8 = 12(1-r) \implies 1 - r = \frac{2}{3} \implies r = \frac{1}{3}.$$

Check: $|r| = \frac{1}{3} < 1$, so convergence is valid.

(b) $u_5 = ar^4 = 8\left(\frac{1}{3}\right)^4 = 8 \times \frac{1}{81} = \frac{8}{81}$.

(c) $S_n = \dfrac{a(1-r^n)}{1-r} = \dfrac{8\left(1 - (\frac{1}{3})^n\right)}{1 - \frac{1}{3}} = \dfrac{8\left(1 - (\frac{1}{3})^n\right)}{\frac{2}{3}} = 12\left[1 - (\tfrac{1}{3})^n\right]$.

We require $S_n > 11.5$:

$$12\left[1 - (\tfrac{1}{3})^n\right] > 11.5$$
$$1 - (\tfrac{1}{3})^n > \frac{11.5}{12} = \frac{23}{24}$$
$$(\tfrac{1}{3})^n < 1 - \frac{23}{24} = \frac{1}{24}.$$

Using GC or logs:

$$n\ln(\tfrac{1}{3}) < \ln(\tfrac{1}{24}) \implies n > \frac{\ln(1/24)}{\ln(1/3)} \approx 2.89.$$

Since n is an integer, $n \geq 3$. *Verify:* $S_2 = 12\left[1 - (\frac{1}{3})^2\right] = 12 \times \frac{8}{9} = 10.67$ (too small), $S_3 = 12\left[1 - (\frac{1}{3})^3\right] = 12 \times \frac{26}{27} \approx 11.56 > 11.5$. So least number is 3.

Recurrence Relation Problem

Example Recurrence — Conjecture and Prove

A sequence is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 2$ for $n \geq 1$.

(a) Find u_2, u_3 and u_4.

(b) Conjecture a formula for u_n in terms of n.

(c) Prove your formula by induction.

Solution.

(a) $u_2 = 3(2) + 2 = 8$,
$u_3 = 3(8) + 2 = 26$,
$u_4 = 3(26) + 2 = 80$.

(b) Pattern: $u_1 = 2 = 3^1 - 1$, $u_2 = 8 = 3^2 - 1$, $u_3 = 26 = 3^3 - 1$, $u_4 = 80 = 3^4 - 1$.
Conjecture: $u_n = 3^n - 1$.

(c) **Proof by induction.**

- *Base case:* $n = 1$, LHS $= u_1 = 2$, RHS $= 3^1 - 1 = 2$. True.
- *Inductive hypothesis:* Assume $u_k = 3^k - 1$ for some $k \geq 1$.
- *Inductive step:*

$$\begin{aligned} u_{k+1} &= 3u_k + 2 \quad \text{(by recurrence)} \\ &= 3(3^k - 1) + 2 \quad \text{(by hypothesis)} \\ &= 3^{k+1} - 3 + 2 \\ &= 3^{k+1} - 1. \end{aligned}$$

  Hence the formula holds for $n = k + 1$.
- By mathematical induction, $u_n = 3^n - 1$ for all $n \in \mathbb{Z}^+$.

Exam Tip GC for recurrence

Use the GC's sequence mode (e.g. TI-84: MODE → SEQ, then enter $u(n) = 3u(n-1) + 2$ with $u(1) = 2$). The GC will generate terms instantly, which helps with spotting the pattern. For the proof, however, you must write out the full induction by hand.

Method of Differences (Telescoping Series)

Example Method of Differences

Express $\dfrac{2}{r^2 + 2r}$ in partial fractions. Hence find $\displaystyle\sum_{r=1}^{n} \frac{2}{r^2 + 2r}$ in terms of n, and deduce the sum to infinity.

Solution.

Factorise: $r^2 + 2r = r(r + 2)$. Write:

$$\frac{2}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2}.$$

Multiply through: $2 = A(r+2) + Br$.
Let $r = 0$: $2 = 2A \implies A = 1$.
Let $r = -2$: $2 = -2B \implies B = -1$.
Hence $\dfrac{2}{r^2+2r} = \dfrac{1}{r} - \dfrac{1}{r+2}$.

Now sum:

$$\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+2}\right) = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Cancelling:
Positive terms remaining: $1 + \frac{1}{2}$.
Negative terms remaining: $-\frac{1}{n+1} - \frac{1}{n+2}$.

Thus:

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}.$$

As $n \to \infty$, $\frac{1}{n+1} \to 0$ and $\frac{1}{n+2} \to 0$, so:

$$S_\infty = \frac{3}{2}.$$

Exam Tip Telescoping pattern

For method of differences, write out the first **three** terms and the last **three** terms of the expanded sum. Circle or highlight the terms that cancel — this prevents careless errors. Always check that the number of surviving terms is exactly 2× (the offset in the denominator).

Sigma Notation Evaluation

Example Sigma Notation

Evaluate $\displaystyle\sum_{r=1}^{30}(2r-1)^2$.

Solution.

First expand:

$$(2r-1)^2 = 4r^2 - 4r + 1.$$

Hence:

$$\begin{aligned}\sum_{r=1}^{30}(2r-1)^2 &= \sum_{r=1}^{30}(4r^2 - 4r + 1) \\ &= 4\sum_{r=1}^{30} r^2 - 4\sum_{r=1}^{30} r + \sum_{r=1}^{30} 1.\end{aligned}$$

Use standard results:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2},$$
$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{r=1}^{n} 1 = n.$$

With $n = 30$:

$$\sum r^2 = \frac{30 \times 31 \times 61}{6} = \frac{56730}{6} = 9455,$$
$$\sum r = \frac{30 \times 31}{2} = 465,$$
$$\sum 1 = 30.$$

Therefore:

$$4(9455) - 4(465) + 30 = 37820 - 1860 + 30 = 35990.$$

Warning Formula errors

The most common mistake in sigma notation questions is using the wrong formula for $\sum r^2$. Remember: $\sum r^2 = \frac{n(n+1)(2n+1)}{6}$. A good check: for $n = 1$, this gives $\frac{1 \cdot 2 \cdot 3}{6} = 1$, which is correct.

Financial Maths Application

Example Loan Repayment — Geometric Series

Darryl takes a loan of \$50,000 at an annual interest rate of 3.6%, compounded monthly. He repays \$800 at the end of each month.

(a) Show that the outstanding amount after n months is given by

$$A_n = 50000(1.003)^n - 800\frac{(1.003)^n - 1}{0.003}.$$

(b) Find the number of months needed to fully repay the loan.

(c) Find the total interest paid.

Solution.

Monthly interest rate: $\dfrac{3.6\%}{12} = 0.3\% = 0.003.$

(a) Month 1: $A_1 = 50000(1.003) - 800.$
Month 2: $A_2 = [50000(1.003) - 800](1.003) - 800 = 50000(1.003)^2 - 800(1.003) - 800.$
Month 3:

$$A_3 = 50000(1.003)^3 - 800(1.003)^2 - 800(1.003) - 800.$$

After n months:

$$A_n = 50000(1.003)^n - 800\left[1 + (1.003) + (1.003)^2 + \cdots + (1.003)^{n-1}\right].$$

The bracket is a GP: $a = 1$, $r = 1.003$, n terms.

$$S_n = \frac{1[(1.003)^n - 1]}{1.003 - 1} = \frac{(1.003)^n - 1}{0.003}.$$

Therefore:

$$A_n = 50000(1.003)^n - 800\frac{(1.003)^n - 1}{0.003},$$

as required.

(b) When loan is repaid: $A_n = 0$.

$$\begin{aligned} 50000(1.003)^n &= 800\frac{(1.003)^n - 1}{0.003} \\ 50000(1.003)^n \times 0.003 &= 800\left[(1.003)^n - 1\right] \\ 150(1.003)^n &= 800(1.003)^n - 800 \\ 800 &= 650(1.003)^n \\ (1.003)^n &= \frac{800}{650} = \frac{16}{13} \approx 1.23077. \end{aligned}$$

Taking logs:

$$n \ln(1.003) = \ln(1.23077) \implies n = \frac{\ln(1.23077)}{\ln(1.003)} \approx 69.6.$$

Hence $n = 70$ months (since repayment at end of month 70 clears the loan).

(c) Total repaid $= 70 \times 800 = \$56,000$.
Total interest $= 56000 - 50000 = \$6,000$.

Exam Tip Interest rate trap

The most common mistake in financial maths is using the annual rate instead of the monthly rate. Always divide by 12 for monthly compounding. Also, check whether payment is at the **end** (ordinary annuity) or **beginning** (annuity due) of each period — this changes the formula.

Answering Techniques

Handling 'Hence' Questions

The word **hence** signals a dependency chain. Marks are awarded for using the previous result.

Exam Tip How to use 'hence'

1. Explicitly state: "From part (a), we have . . . "
2. Quote the relevant result before manipulating it.
3. **Never** re-derive the previous result — this wastes time and may lose method marks if the examiner can't see you building on it.

Example Hence Technique

Part (a) asked you to show $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}$.

Part (b): Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$.

Correct approach:

$$\begin{aligned}\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} &= \text{Let } s = r+1, \text{ then as } r = 1 \to n,\ s = 2 \to n+1. \\ &= \sum_{s=2}^{n+1} \frac{1}{s(s+1)} \\ &= \left(1 - \frac{1}{n+2}\right) - \left(1 - \frac{1}{2}\right) \quad \text{(using result from (a) at limits)} \\ &= \frac{1}{2} - \frac{1}{n+2}.\end{aligned}$$

Always show the index shift explicitly.

GC Techniques for Recurrence Relations

- **TI-84 Plus / TI-Nspire:** Use Sequence mode (MODE → SEQ). Enter $u(n)$ = [expression], $u(1)$ = [initial]. The table view generates $u_1, u_2, \ldots, u_{20}$ instantly.
- **Spotting convergence:** Generate 10–15 terms and observe whether values approach a limit. If the sequence converges, the limit L satisfies $L = f(L)$ for the recurrence $u_{n+1} = f(u_n)$.
- **Checking induction:** Use the GC to compute the conjectured closed-form for $n = 1$ through $n = 10$ and compare with the recurrence-generated values. If they match for 5+ terms, your conjecture is likely correct.
- **Iterative refinement:** For sequences defined by $u_{n+1} = \sqrt{u_n + 1}$, use the GC to apply the recurrence repeatedly and observe the fixed point.

Warning GC on Paper 1

The 9758 Paper 1 allows use of a GC. However, for proof questions and "show that" parts, you must present algebraic reasoning. Writing "By GC, $u_5 = 3.15$" without showing the recurrence application is insufficient for method marks.

Proving Convergence

Definition Convergence of a Sequence

A sequence $\{u_n\}$ converges to a limit L if for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|u_n - L| < \epsilon$ for all $n > N$. For H2 Math, you are expected to handle convergence of:

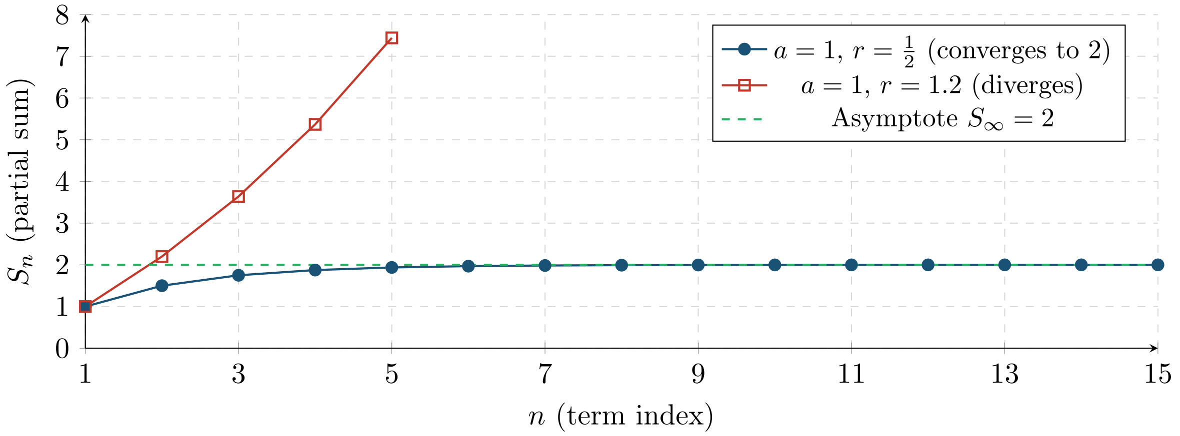
- Geometric series: converges iff $|r| < 1$.
- Recurrence-defined sequences: find fixed point $L = f(L)$, then prove monotonic bounded behaviour.
- Telescoping series: show that $S_n \to L$ as $n \to \infty$ by evaluating the limit of the closed-form expression.

Exam Tip Convergence check

For a GP, always state the condition $|r| < 1$ before using $S_\infty = a/(1-r)$. Writing S_∞ without this check loses a mark. For non-GP series, use the nth term test: if $\lim_{n\to\infty} u_n \neq 0$, the series diverges.

Convergence Visualisation

The diagram below shows the behaviour of a geometric series with $a = 1$, $r = \frac{1}{2}$ converging to $S_\infty = 2$, compared with a divergent series ($r = 1.2$):



Timing Guide

For a typical 3-hour Paper 1 (100 marks total):

Question Type	Recommended Time	Marks
Simple AP/GP (find a, d/r, S_n)	6–8 minutes	4–6
Sum to infinity with condition	4–5 minutes	3–4
Method of differences (full working)	10–12 minutes	6–8
Sigma notation with standard sums	5–7 minutes	4–5
Recurrence + induction proof	10–12 minutes	6–7
Financial maths application	8–10 minutes	5–6

Exam Tip Exam strategy

- If a Sequences and Series question appears early (Q1–Q3), aim to complete it within the time guide above. These are typically standard and should be high-confidence marks.
- For the induction proof part, allocate extra time to check the base case and the algebraic manipulation in the inductive step — this is where most students drop marks.
- If stuck on a "hence" part for more than 3 minutes, move on and return later. The link between parts is sometimes non-obvious and may become clearer after attempting other questions.

Common Errors from Examiner Reports

Based on Cambridge 9758 Examiner Reports (2018–2024), the following errors recur frequently:

Top 10 Mistakes

1. **Using n instead of $n-1$ in u_n:** $u_n = a + (n-1)d$, **not** $a + nd$. Always check with $n = 1$.
2. **Forgetting the convergence condition:** Writing $S_\infty = \dfrac{a}{1-r}$ without stating $|r| < 1$ loses the first method mark.
3. **Mishandling the nth term of a GP:** $u_n = ar^{n-1}$, not ar^n. Double-check with $n = 1$: $u_1 = ar^0 = a$ ✓.
4. **Sigma notation limit errors:** Confusing $\sum_{r=1}^{n}$ with $\sum_{r=0}^{n-1}$. When shifting indices, always write out the first and last terms to verify.
5. **Partial fractions sign errors:** In method of differences, a sign mistake in A or B propagates through the entire cancellation. Check by recombining the fractions.
6. **Induction: missing the inductive hypothesis statement:** You must explicitly state "Assume true for $n = k$" before the inductive step. Simply writing the algebra without this statement loses a method mark.
7. **Induction: forgetting the conclusion:** Always end with "Hence by mathematical induction, the statement is true for all positive integers n."
8. **Financial maths: annual vs. monthly rate:** As noted above, this is the single most common error in applications questions. **Always** divide the annual rate by the number of compounding periods.
9. **GC over-reliance:** Using the GC to compute S_n for a large n without showing the formula earns zero marks on "show that" or "find in terms of n" questions.
10. **Rounding prematurely:** In financial maths, using rounded intermediate values (e.g. 1.003 as 1.003 without sufficient precision) leads to an answer that is off by several months. Store full precision in the GC, or use exact fractions where possible.

Warning Common arithmetic errors

- $\frac{n}{2}[2a + (n-1)d]$: Ensure the factor $2a$ is correct — students often write a instead.
- $\frac{a(1-r^n)}{1-r}$ vs $\frac{a(r^n-1)}{r-1}$: These are equivalent, but sign errors occur when the denominator is negative. Use the form with $(1 - r^n)$ when $|r| < 1$ to avoid confusion.
- Telescoping: Students often miscount the number of surviving terms. For $\sum_{r=1}^{n} \frac{1}{r(r+2)}$, after partial fractions, four terms survive (1, $\frac{1}{2}$, $-\frac{1}{n+1}$, $-\frac{1}{n+2}$), not two.

Last-Minute Checklist Before the Exam

- Can you write all four standard formulas from memory?

$$u_n^{(\mathrm{AP})} = a + (n-1)d, \qquad S_n^{(\mathrm{AP})} = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l),$$
$$u_n^{(\mathrm{GP})} = ar^{n-1}, \qquad S_n^{(\mathrm{GP})} = \frac{a(1-r^n)}{1-r}, \quad S_\infty = \frac{a}{1-r} \text{ if } |r| < 1.$$

- Can you recall $\sum r = \frac{n(n+1)}{2}$, $\sum r^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum r^3 = \frac{n^2(n+1)^2}{4}$?
- Do you know the induction template (base case → hypothesis → inductive step → conclusion)?
- Can you set up a loan repayment or savings annuity as a geometric series?

Exam Tip Final advice

Sequences and Series is one of the most predictable topics in 9758 Paper 1. The question formats change very little year-to-year. Master the **six question templates** in Section 3, and you can confidently secure 8–12 marks on every paper.