H2 Mathematics (9758)

Geometric Derivations: Standard Graphs

Why the formulas look the way they do

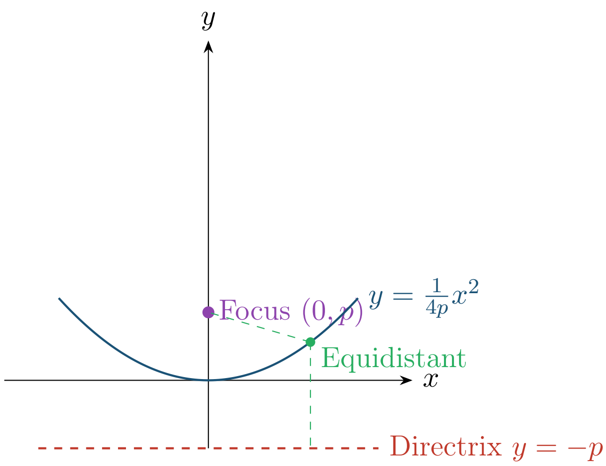
Introduction

Every standard graph shape comes from a simple geometric definition. Understanding *why* the equations take the form they do eliminates the need to memorise — you can rederive them from first principles.

The Parabola: $y = ax^2$

Geometric Definition

A parabola is the set of all points **equidistant from a fixed point (the focus) and a fixed line (the directrix)**.



Derivation

Place the focus at $(0, p)$ and the directrix at $y = -p$. Take any point (x, y) on the parabola.

Distance to focus:

$$\sqrt{(x-0)^2 + (y-p)^2}$$

Distance to directrix (perpendicular = vertical distance):

$$|y + p|$$

Equate them:

$$\sqrt{x^2 + (y-p)^2} = |y + p|$$

Square both sides:

$$x^2 + (y-p)^2 = (y+p)^2$$

Expand:

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

Cancel y^2 and p^2:

$$x^2 = 4py \quad \Rightarrow \quad \boxed{y = \frac{1}{4p}x^2}$$

Let $a = 1/(4p)$, giving $y = ax^2$.

Why the Vertex is at $(0, 0)$

The vertex is the **midpoint** between the focus and the directrix along the axis of symmetry. Focus at $(0, p)$, directrix at $y = -p$, so the vertex is at $(0, 0)$ — the point on the parabola closest to both.

Why $y = a(x - h)^2 + k$ has Vertex at (h, k)

If you translate the entire focus-directrix arrangement: focus $\to (h, k + p)$, directrix $\to y = k - p$, then repeating the derivation gives:

$$y = \frac{1}{4p}(x - h)^2 + k = a(x - h)^2 + k$$

Derivative proof: $dy/dx = 2a(x - h)$. Setting this to 0 gives $x = h$, and $y(h) = k$.

Sideways Parabola: $y^2 = bx$

Swap the roles of x and y — focus on the x-axis, directrix vertical. The same logic gives:

$$y^2 = 4px = bx$$

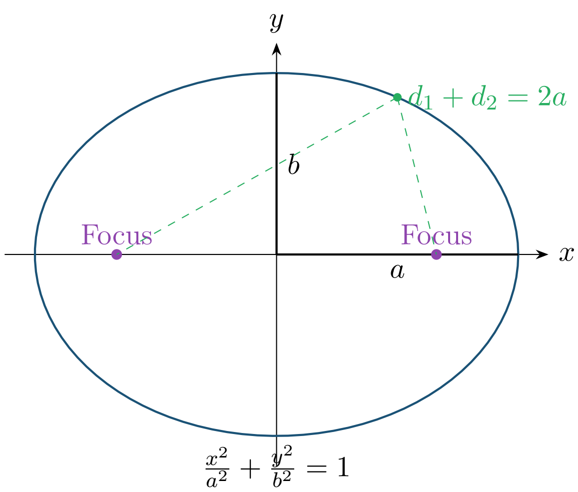
Key Takeaway

- $y = ax^2$ is just a compact way of writing "a point whose distance to the focus equals its distance to the directrix"
- The coefficient a encodes how far apart the focus and directrix are ($p = 1/4a$)
- Bigger p (smaller a) means a "wider" parabola — the focus is farther from the vertex

The Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Geometric Definition

An ellipse is the set of all points where the **sum** of distances to two fixed points (the foci) is **constant**.



Derivation

Place foci at $(\pm c, 0)$. Let $2a$ be the constant sum of distances. For any point (x, y):

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

Isolate one square root and square:

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$
$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

Expand the left:

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

Cancel $x^2+c^2+y^2$ from both sides:

$$2cx+c^2=4a^2-4a\sqrt{(x-c)^2+y^2}-2cx+c^2$$
$$4cx=4a^2-4a\sqrt{(x-c)^2+y^2}$$

Divide by 4:

$$cx=a^2-a\sqrt{(x-c)^2+y^2}$$
$$a\sqrt{(x-c)^2+y^2}=a^2-cx$$

Square again:

$$a^2\left[(x-c)^2+y^2\right]=a^4-2a^2cx+c^2x^2$$
$$a^2(x^2-2cx+c^2+y^2)=a^4-2a^2cx+c^2x^2$$
$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

The $-2a^2cx$ terms cancel:

$$a^2x^2+a^2c^2+a^2y^2=a^4+c^2x^2$$

Group x^2 terms:

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$
$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

Introducing b

Define $b^2=a^2-c^2$. This is the semi-minor axis length (check it: when $x=0$, $y^2=b^2$). Then:

$$b^2x^2+a^2y^2=a^2b^2$$

Divide through by a^2b^2:

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$

What a and b Mean

- a is the **semi-major axis** (half the longer diameter)
- b is the **semi-minor axis** (half the shorter diameter)
- $c=\sqrt{a^2-b^2}$ is the distance from centre to each focus
- When $x=0$: $y=\pm b$ (y-intercepts)
- When $y=0$: $x=\pm a$ (x-intercepts)
- If $a>b$, the ellipse is wider than tall
- If $a<b$, the roles swap — foci are on the y-axis

Key Takeaway

- The "+" sign in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ comes from the sum-of-distances definition
- a is half the total path length ($2a$ is the constant sum)
- $c < a$ for an ellipse, so $b^2 = a^2 - c^2 > 0$ — everything is well-defined

The Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Geometric Definition

A hyperbola is the set of all points where the **difference** of distances to two fixed points (the foci) is **constant**.

Derivation

Same setup as the ellipse — foci at $(\pm c, 0)$ — but now:

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$$

Follow the *exact same* algebraic steps as the ellipse. The only change is the sign: instead of a $+$ between the two squared terms, we get a $-$.

The algebra works out to:

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

For a hyperbola, $c > a$ (the foci are farther apart than the constant difference allows), so $c^2 - a^2 > 0$. Define $b^2 = c^2 - a^2$:

$$b^2x^2 - a^2y^2 = a^2b^2$$

Divide through:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Where the Asymptotes Come From

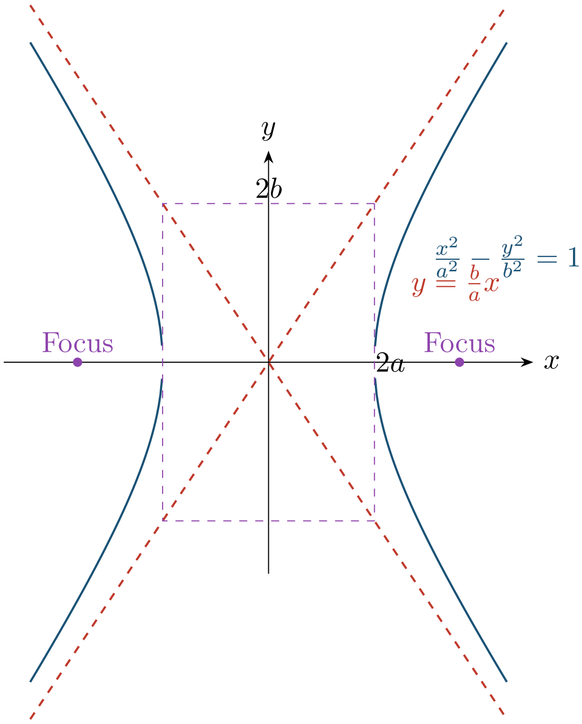
As x becomes very large, the 1 on the right is negligible:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \approx 0$$

$$\frac{y^2}{b^2} \approx \frac{x^2}{a^2} \quad \Rightarrow \quad |y| \approx \frac{b}{a}|x|$$

The hyperbola approaches the lines $y = \pm\frac{b}{a}x$ as $x \to \pm\infty$ but never reaches them.

The a-b-c Box



- The box has width $2a$ (from $x = -a$ to $x = a$), height $2b$
- The **asymptotes** are the diagonals of this box
- $c = \sqrt{a^2 + b^2}$ is the distance from centre to each focus
- **Contrast with ellipse**: $c^2 = a^2 + b^2$ for hyperbola vs $c^2 = a^2 - b^2$ for ellipse. This is because we defined $b^2 = c^2 - a^2$ here, whereas for the ellipse $b^2 = a^2 - c^2$.

Key Takeaway

- The "$-$" sign reflects the **difference** definition (vs the sum for ellipse)
- The asymptotes are not arbitrary — they're the natural behaviour as x grows large, where the constant 1 becomes negligible
- The box construction gives you asymptotes instantly: draw the rectangle with sides $2a$ and $2b$, then diagonals

Rational Functions: Asymptotes

Reciprocal Linear: $y = \dfrac{ax + b}{cx + d}$

Vertical asymptote: A fraction can't have denominator 0. So:

$$cx + d = 0 \quad \Rightarrow \quad x = -\frac{d}{c}$$

As x approaches this value, the denominator $\to 0$ while the numerator is finite, so $|y| \to \infty$ — hence a vertical asymptote.

Horizontal asymptote: Look at what happens as $x \to \pm\infty$:

$$y = \frac{ax + b}{cx + d} = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}$$

As $x \to \pm\infty$, $\frac{b}{x} \to 0$ and $\frac{d}{x} \to 0$, so:

$$y \to \frac{a}{c}$$

Intuitively: for very large x, the constants b and d become irrelevant compared to ax and cx, so the ratio approaches a/c.

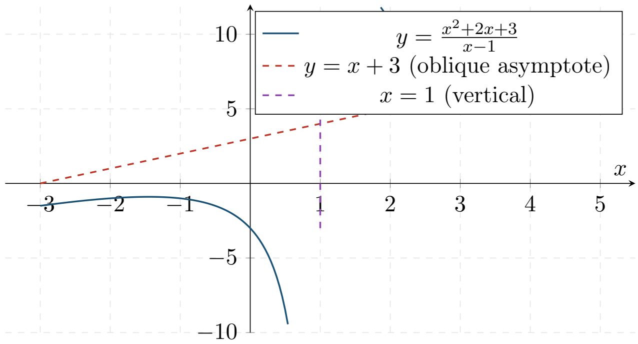
Improper Rational: $y = \dfrac{ax^2 + bx + c}{dx + e}$

The numerator has degree 2, denominator degree 1. Divide:

$$\frac{ax^2 + bx + c}{dx + e} = (px + q) + \frac{r}{dx + e}$$

The quotient $px + q$ is a **linear** function. As $x \to \pm\infty$, the remainder $\frac{r}{dx+e} \to 0$, so the graph approaches the **oblique asymptote** $y = px + q$.

Why polynomial division works: It rewrites the fraction as a polynomial plus a proper fraction. The proper fraction vanishes at infinity, leaving the polynomial as the asymptotic shape.



Asymptote Summary

Type	When
Vertical	Denominator $= 0$, numerator $\neq 0$
Horizontal	Degree numerator $<$ degree denominator $\Rightarrow y = 0$
	Degree numerator $=$ degree denominator $\Rightarrow y =$ ratio of leading coefficients
Oblique	Degree numerator $=$ degree denominator $+1$
None of the above	Degree numerator $\geq$ degree denominator $+2$ (uncommon at A-Level)

Summary: Definitions → Equations

Shape	Geometric Definition
Parabola	Equidistant from focus and directrix
Ellipse	Sum of distances to two foci is constant
Hyperbola	Difference of distances to two foci is constant
Rational function	Algebraic ratio — asymptotes from limits

Equation	**Asymptotes / Key features**
$y = ax^2$	Vertex at $(0, 0)$; no asymptotes
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	Closed curve; x-intercepts $\pm a$, y-intercepts $\pm b$
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$y = \pm\frac{b}{a}x$ (from large-x behaviour)
$y = \dfrac{ax+b}{cx+d}$	$x = -d/c$, $y = a/c$
$y = \dfrac{ax^2+bx+c}{dx+e}$	$x = -e/d$, $y =$ quotient from division

The unifying theme: **every standard graph equation comes from a simple geometric condition**. When you forget a formula, go back to the definition and derive it.