H2 Mathematics (9758)

Condensed Revision: Sequences & Series + Graphs & Transformations

A-Level 2027 Syllabus

Sequences and Series

Must-Know Formulas (Memorise These)

Concept	Formula
AP nth term	$u_n = a + (n-1)d$
AP Sum to n	$S_n = \frac{n}{2}\big(2a + (n-1)d\big) = \frac{n}{2}(a + \ell)$
GP nth term	$u_n = ar^{n-1}$
GP Sum to n	$S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$
GP Sum to ∞	$S_\infty = \frac{a}{1-r} \quad$ if $\lvert r\rvert < 1$
Sigma notation	$\sum_{r=1}^{n} u_r = S_n$
u_n from S_n	$u_n = S_n - S_{n-1}$ for $n \geq 2$, $u_1 = S_1$

Derivations — How the Formulas Come About

Derivation AP Sum Formula

Write S_n forwards then backwards and add:

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) \\ S_n &= (a+(n-1)d) + (a+(n-2)d) + \cdots + a \\ \hline 2S_n &= n\big[2a + (n-1)d\big] \\ S_n &= \frac{n}{2}\big[2a + (n-1)d\big] \end{aligned}$$

Key insight: Each column sums to $2a + (n-1)d$. There are n columns.

Derivation GP Sum Formula

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$.
Multiply by r:

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Subtract (most terms cancel):

$$S_n - rS_n = a - ar^n \quad \Rightarrow \quad S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}} \qquad (r \neq 1)$$

If $|r| < 1$, as $n \to \infty$, $r^n \to 0$, so:

$$\boxed{S_\infty = \frac{a}{1-r}}$$

Method of Differences (Telescoping)

A commonly tested technique. The idea: split a term into a difference, then most terms cancel.

Example Standard Method of Differences

Find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)}$.
Solution: First decompose via partial fractions:

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

Now write out the sum:

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

All interior terms cancel. Only the first and last survive:

$$\boxed{\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}}$$

For $n \to \infty$: $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1$ (converges).

Warning Common Pitfall

Method of differences only works if the general term is *exactly* expressible as a difference. Always verify the partial fraction decomposition first. Watch for off-by-one errors in the last term.

Recurrence Relations

A sequence can be defined by $u_{n+1} = f(u_n)$.

- Use GC to generate terms: enter u_n formula mode, set $u(1) = \text{initial}$, $u(n) = f(u(n-1))$.
- To find a closed form, compute the first few terms and look for a pattern (AP, GP, or other).

- For convergence of $u_{n+1} = f(u_n)$, find fixed points $x = f(x)$. The sequence converges if $|f'(x)| < 1$ near the fixed point (contraction mapping).

Exam Tip GC for Recurrence Relations

TI-84 Plus CE:
1. Mode → Seq
2. y= → enter nMin $= 1$, $u(n) =$ expression, $u(n\text{Min}) =$ initial value
3. Graph or Table to see terms

TI-Nspire:
1. Add Graphs & Geometry
2. Entry → Sequence
3. Enter expression and initial condition

Convergence of Series

A series $\sum u_n$ **converges** if its partial sums S_n approach a finite limit.

- **Necessary** (but not sufficient) condition: $\lim_{n\to\infty} u_n = 0$
- Geometric series $\sum ar^{n-1}$ converges $\iff |r| < 1$
- Harmonic series $\sum \frac{1}{n}$ diverges even though $u_n \to 0$ (important counterexample!)

Commonly Tested Question Types

1. **AP/GP word problems** — Identify the pattern, find a and d or a and r, then use formulas.
2. **Sum to infinity** — Check $|r| < 1$ first, then apply $S_\infty = a/(1-r)$.
3. **Sigma notation with method of differences** — Partial fractions → telescope → evaluate.
4. **Recurrence + conjecture** — Find first terms, conjecture u_n formula, prove by induction.
5. **Financial maths** — Loan repayment as geometric series. Total $= P(1+i)^n$ for compound interest; monthly payment $= \frac{Pi(1+i)^n}{(1+i)^n-1}$.
6. **Finding u_n from S_n** — $u_n = S_n - S_{n-1}$.

Exam Tip AP/GP Identification

Given a sequence $p, q, r, \ldots$:

- Check if $q - p = r - q$ (constant difference ⇒ AP)
- Check if $\frac{q}{p} = \frac{r}{q}$ (constant ratio ⇒ GP)
- If neither, try second difference or look for recurrence pattern

Graphs and Transformations

Standard Graphs — Know These Shapes

Equation	Key features
$y = ax^2$	Parabola, vertex at $(0,0)$, axis of symmetry $x = 0$
$y^2 = bx$	Sideways parabola, vertex $(0,0)$, opens right if $b > 0$
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse, centre $(0,0)$, x-intercepts $\pm a$, y-intercepts $\pm b$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Hyperbola, asymptotes $y = \pm\frac{b}{a}x$, x-intercepts $\pm a$
$y = \frac{ax+b}{cx+d}$	Reciprocal linear. Vertical asymptote $x = -\frac{d}{c}$, horizontal $y = \frac{a}{c}$
$y = \frac{ax^2+bx+c}{dx+e}$	Improper rational. Polynomial division → linear + proper fraction. Oblique asymptote from quotient.

Graph Sketching Checklist

1. **Domain:** Any restrictions? (e.g. $x > -1$ for $\ln(x+1)$, $x \neq -d/c$ for rational functions)
2. **Asymptotes:** Find vertical (denominator $= 0$), horizontal (degree check), oblique (polynomial division)
3. **Intercepts:** x-intercepts from $y = 0$, y-intercept from $x = 0$
4. **Turning points:** Differentiate and solve $dy/dx = 0$
5. **Behaviour near asymptotes:** Check limits as $x \to \text{asymptote}^{\pm}$
6. **Axes of symmetry:** Even function ($f(-x) = f(x)$) $\Rightarrow$ symmetry in y-axis. Odd function ($f(-x) = -f(x)$) $\Rightarrow$ rotational symmetry.

Exam Tip Asymptote Types

- **Vertical:** $x = c$ where denominator $= 0$ and numerator $\neq 0$
- **Horizontal:** $y = c$ where $\lim_{x\to\pm\infty} f(x) = c$. Degree of numerator $<$ degree of denominator $\Rightarrow y = 0$; equal degrees $\Rightarrow y =$ ratio of leading coefficients
- **Oblique (slant):** Degree numerator $=$ degree denominator $+1$. Do polynomial division; the quotient line is the asymptote.

Always draw asymptotes as **dashed lines** and label their equations.

Transformations — Order Matters

Transformation	Effect on $y = f(x)$
$y = f(x) + a$	Vertical translation: shift up by a ($a > 0$)
$y = f(x + a)$	Horizontal translation: shift **left** by a ($a > 0$)
$y = af(x)$	Vertical stretch: scale factor a parallel to y-axis
$y = f(ax)$	Horizontal compression: scale factor $1/a$ parallel to x-axis
$y = -f(x)$	Reflection in x-axis
$y = f(-x)$	Reflection in y-axis

Warning Order of Transformations

Apply transformations in the **reverse** order for horizontal changes.
For $y = 3f(2x-4)+1$:

1. Factor inside bracket: $2x - 4 = 2(x-2)$
2. Apply in order: $f(x) \to f(x-2)$ (shift right 2) $\to f(2(x-2))$ (compress horizontally $\times\frac{1}{2}$) $\to 3f(2x-4)$ (stretch vertically $\times 3$) $\to 3f(2x-4)+1$ (shift up 1)

Golden rule: Inside the function $(ax+b)$, operations on x are *reverse-intuitive*. $f(x+2)$ shifts *left*, $f(2x)$ compresses by $\frac{1}{2}$.

Modulus Functions

$y = |f(x)|$**:** Reflect the portion below the x-axis upward.

- Where $f(x) \geq 0$: $|f(x)| = f(x)$ (unchanged)
- Where $f(x) < 0$: $|f(x)| = -f(x)$ (reflected in x-axis)
- The graph is always ≥ 0

$y = f(|x|)$**:** Reflect the right-hand side to the left.

- For $x \geq 0$: $f(|x|) = f(x)$ (same as original right half)
- For $x < 0$: $f(|x|) = f(-x)$ (reflect the right half across y-axis)
- The graph is **even**: symmetric about y-axis

Reciprocal Graphs: $y = 1/f(x)$

Key relationships between $y = f(x)$ and $y = 1/f(x)$:

Where $f(x)$ is...	**$1/f(x)$ will be...**
Zero ($f(x) = 0$)	Vertical asymptote
Large $\|f(x)\|$	Close to 0 (horizontal asymptote at $y = 0$)
Maximum (positive)	Minimum (positive)
Minimum (positive)	Maximum (positive)
Vertical asymptote	Zero ($1/f(x) = 0$)
$f(x) = 1$	$1/f(x) = 1$ (fixed point)
$f(x) = -1$	$1/f(x) = -1$ (fixed point)

Exam Tip Sketching $1/f(x)$

1. Draw asymptotes where $f(x) = 0$ (vertical lines)
2. Find where $f(x) = \pm 1$ — these are fixed points
3. Where $f(x) \to \infty$, $1/f(x) \to 0^+$; where $f(x) \to -\infty$, $1/f(x) \to 0^-$
4. Turning points of f map to turning points of $1/f$ (but reciprocal in value)

Parametric Equations

A curve defined by $x = f(t)$, $y = g(t)$.

- To sketch: tabulate (x, y) for several t values, plot points in order of increasing t

- Direction of curve: indicate with arrows as t increases
- Gradient: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ (Chain rule)
- Converting to cartesian: eliminate t using substitution or trigonometric identities

Example Parametric to Cartesian

Given $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$.
Solution: $\cos t = \frac{x}{2}$, $\sin t = \frac{y}{2}$. Using $\cos^2 t + \sin^2 t = 1$:

$$\frac{x^2}{4} + \frac{y^2}{4} = 1 \quad \Rightarrow \quad x^2 + y^2 = 4$$

This is a circle centre $(0, 0)$ radius 2. As t increases from 0 to 2π, the circle is traced anticlockwise.

Commonly Tested Question Types

1. **Sketch a rational function** (4–6 marks) — Find asymptotes, intercepts, turning points. Show all on graph.
2. **Apply sequence of transformations** (3–5 marks) — Given $y = f(x)$ and transformed equation, describe or sketch the result.
3. **Relate $y = |f(x)|$, $y = f(|x|)$, $y = 1/f(x)$** (4–5 marks) — Given $f(x)$, sketch the transformed version.
4. **Combined transformations + function notation** — Find image of a point, or equation of asymptotes after transformation.
5. **Parametric to cartesian + sketch** (3–5 marks) — Eliminate parameter, identify standard curve, sketch with direction.
6. **Graphical solution of equations** (3 marks) — Sketch two curves, find intersection point(s).

Exam Tip Mark-Saving Habits

- Always show asymptote equations with dashed lines — they carry their own mark.
- Label intercepts *exactly*: $(\frac{1}{2}, 0)$ not $(0.5, 0)$.
- For "describe transformation": state type, direction, *and* magnitude for each mark.
- When sketching $|f(x)|$, clearly indicate which part was reflected.
- For rational functions, show the oblique asymptote equation after division.

Quick Reference: Transformation Summary

Equation	**Effect** (Image of (x, y) & asymptotes)
$y = f(x) + a$	Shift up a: $(x, y) \to (x, y + a)$; asymptote $y = c \to y = c + a$
$y = f(x + a)$	Shift left a: $(x, y) \to (x - a, y)$; asymptote $x = c \to x = c - a$
$y = af(x)$	Vertical stretch $\times a$: $(x, y) \to (x, ay)$; asymptote $y = c \to y = ac$
$y = f(ax)$	Horizontal compress $\times \frac{1}{a}$: $(x, y) \to (\frac{x}{a}, y)$; asymptote $x = c \to x = \frac{c}{a}$
$y = -f(x)$	Reflect in x-axis: $(x, y) \to (x, -y)$; $y = c \to y = -c$
$y = f(-x)$	Reflect in y-axis: $(x, y) \to (-x, y)$; $x = c \to x = -c$
$y = \|f(x)\|$	Reflect negative y upward: $(x, y < 0) \to (x, -y)$
$y = f(\|x\|)$	Reflect right half leftwards: $(x > 0, y) \to (-x, y)$
$y = \dfrac{1}{f(x)}$	Zeros of $f \to$ vertical asymptotes; $f \to \infty \Rightarrow 1/f \to 0$

Warning Don't Confuse These

- $y = f(ax)$ vs $y = af(x)$ — first compresses horizontally, second stretches vertically
- $y = f(x + a)$ shifts **left** $(a > 0)$, *not* right
- For $y = f(ax + b)$, factor first: $f(a(x + \frac{b}{a}))$ — then apply shift then stretch (inside-out)