

H2 Mathematics (9758)

Sequences and Series — LECTURE Notes

A-Level 2027 Syllabus

Syllabus Learning Outcomes

By the end of this topic, you should be able to:

1. Understand the **concepts of sequence and series** for both finite and infinite cases.
2. Interpret a sequence as a **function** $y = f(n)$ where $n \in \mathbb{Z}^+$.
3. Recognise and use the **relationship** between the n th term u_n and the sum to n terms S_n :

$$u_n = S_n - S_{n-1}, \quad S_n = \sum_{k=1}^n u_k.$$

4. Find terms of a sequence given by a **formula for the n th term**.
5. Generate terms from a **recurrence relation** $u_{n+1} = f(u_n)$, including use of the graphing calculator (GC).
6. Compute the **sum and difference of two series** using sigma notation.
7. Determine the **convergence** of a series and compute the sum to infinity where applicable.
8. Derive and apply the formula for the n th term and sum of a finite **arithmetic series**.
9. Derive and apply the formula for the n th term and sum of a finite **geometric series**.
10. State and apply the **condition for convergence** of an infinite geometric series.
11. Compute the **sum to infinity** of a convergent geometric series.
12. Use **sigma notation** to represent and manipulate series.
13. Apply the **method of differences** to find sums of series.
14. Work with **recurrence relations** and recognise different types.

Core Concepts

Sequence vs. Series

Definition Sequence

A **sequence** is an ordered list of numbers following a definite pattern or rule. Each number in the sequence is called a **term**. Mathematically, a sequence is a function whose domain is the set of positive integers \mathbb{Z}^+ .

$$u_1, u_2, u_3, \dots, u_n, \dots$$

where u_n denotes the n th term.

Definition Series

A **series** is the sum of the terms of a sequence. If the sequence is $\{u_n\}$, then the corresponding series is:

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

Finite vs. Infinite

- A **finite sequence/series** has a fixed number of terms. For example: 2, 4, 6, 8 (4 terms).
- An **infinite sequence/series** continues indefinitely. For example: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

Sigma Notation

Sigma notation provides a compact way to write sums. The Greek letter Σ (sigma) denotes summation.

$$\sum_{k=1}^n u_k = u_1 + u_2 + u_3 + \cdots + u_n$$

- k is the **index of summation** (also called the dummy variable).
- The lower limit $k = 1$ indicates where the sum starts.
- The upper limit n indicates where the sum ends.
- The index can be any letter: $\sum_{i=1}^n a_i$, $\sum_{j=0}^m b_j$, etc.

Exam Tip Sigma Notation on GC

On the TI-84 Plus CE, use `sum(seq(expression, variable, start, end))`. On the TI-Nspire CX II, use the summation template from the π (calc) menu or type `sum()` directly. Always check at least the first and last terms manually to verify your setup.

Relationship Between u_n and S_n

The sum to n terms S_n is defined as:

$$S_n = \sum_{k=1}^n u_k = u_1 + u_2 + \cdots + u_n$$

The n th term can be recovered from the sum:

$$u_n = S_n - S_{n-1} \quad \text{for } n \geq 2$$

Warning Domain of $u_n = S_n - S_{n-1}$

The formula $u_n = S_n - S_{n-1}$ is valid only for $n \geq 2$. The first term must be found separately: $u_1 = S_1$. A common mistake in exams is to apply this formula for $n = 1$, which would involve S_0 , a quantity that is not defined.

Example Finding u_n from S_n

Given $S_n = n^2 + 3n$, find u_n and write down the first three terms of the sequence.

Solution:

$$u_1 = S_1 = 1^2 + 3(1) = 4$$

For $n \geq 2$:

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= (n^2 + 3n) - [(n-1)^2 + 3(n-1)] \\ &= (n^2 + 3n) - (n^2 - 2n + 1 + 3n - 3) \\ &= (n^2 + 3n) - (n^2 + n - 2) \\ &= 2n + 2 \end{aligned}$$

Check: $u_1 = 2(1) + 2 = 4$, which matches S_1 . So $u_n = 2n + 2$ for all $n \geq 1$. The first three terms are 4, 6, 8.

Arithmetic Sequences and Series

Definition

Definition Arithmetic Sequence

A sequence is **arithmetic** if the difference between consecutive terms is constant. This constant difference is called the **common difference**, denoted by d .

$$u_{n+1} - u_n = d \quad \text{for all } n \geq 1$$

n th Term Formula

If an arithmetic sequence has first term $a = u_1$ and common difference d , then:

$$u_n = a + (n-1)d$$

Derivation n th Term of an Arithmetic Sequence

$$\begin{aligned} u_1 &= a \\ u_2 &= a + d \\ u_3 &= a + 2d \\ &\vdots \\ u_n &= a + (n-1)d \end{aligned}$$

Each step adds one extra d , so after $n - 1$ steps we have added $(n - 1)d$ to the first term.

Sum of n Terms

Derivation Sum of Arithmetic Series

Let $S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$. Write the sum in reverse:

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + a$$

Adding the two expressions term by term:

$$\begin{aligned} 2S_n &= [a + a + (n - 1)d] + [a + d + a + (n - 2)d] + \cdots + [a + (n - 1)d + a] \\ &= [2a + (n - 1)d] + [2a + (n - 1)d] + \cdots + [2a + (n - 1)d] \quad (n \text{ terms}) \\ &= n[2a + (n - 1)d] \end{aligned}$$

Hence:

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + \ell)$$

where $\ell = u_n = a + (n - 1)d$ is the last term.

Exam Tip Arithmetic Series

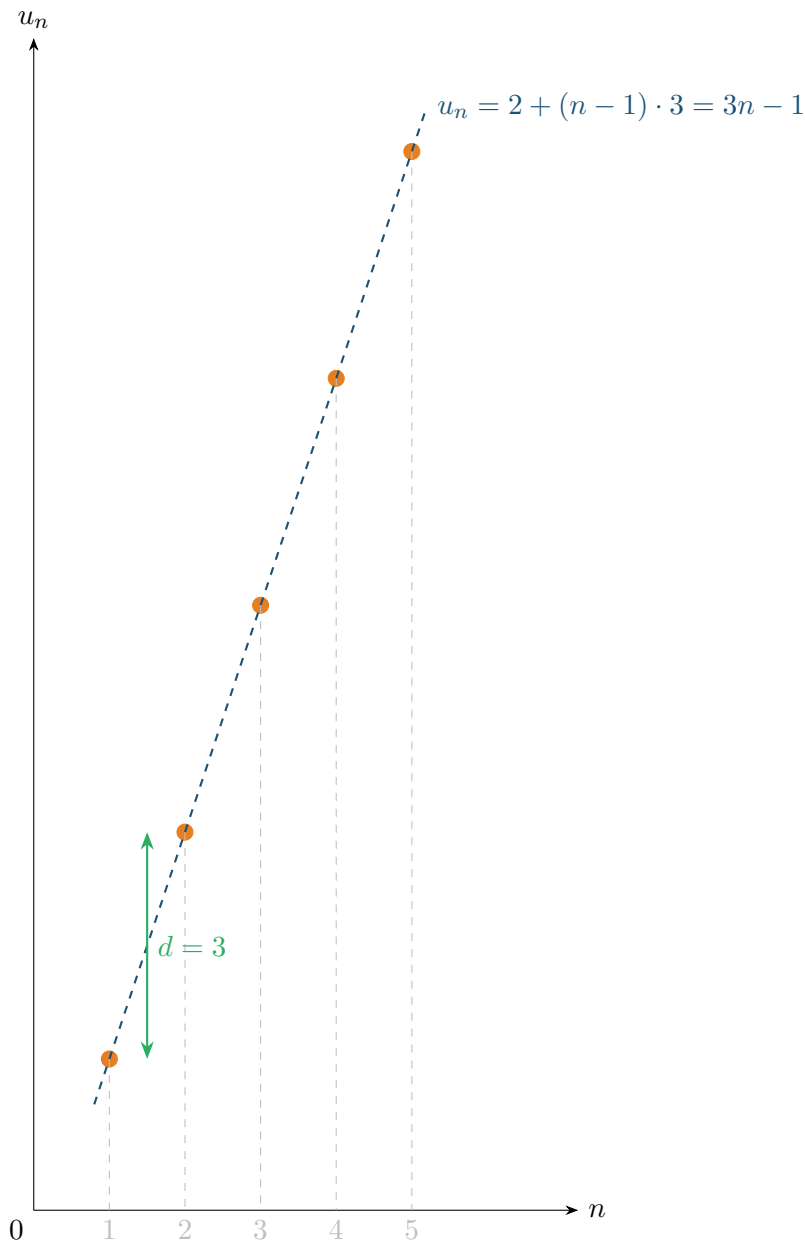
The two equivalent forms are:

- $S_n = \frac{n}{2}[2a + (n - 1)d]$ — use when a and d are known.
- $S_n = \frac{n}{2}(a + \ell)$ — use when the first and last terms are known.

The second form is especially useful in ‘find the sum of the first n integers’ type problems.

Visualising an Arithmetic Progression

The diagram below shows an arithmetic progression with $a = 2$, $d = 3$. Notice the linear growth: the terms lie on a straight line $y = 3n - 1$.



Example Arithmetic Series

The 4th term of an arithmetic progression is 11 and the 9th term is 26.

- (a) Find the first term and the common difference.
 (b) Find the sum of the first 15 terms.

Solution:

$$(a) \quad u_4 = a + 3d = 11$$

$$u_9 = a + 8d = 26$$

Subtracting: $(a + 8d) - (a + 3d) = 26 - 11 \implies 5d = 15 \implies d = 3$. Substituting:
 $a + 9 = 11 \implies a = 2$.

$$(b) \quad S_{15} = \frac{15}{2} [2(2) + (15 - 1)(3)] = \frac{15}{2} (4 + 42) = \frac{15}{2} \times 46 = 345.$$

Geometric Sequences and Series

Definition

Definition Geometric Sequence

A sequence is **geometric** if the ratio of consecutive terms is constant. This constant ratio is called the **common ratio**, denoted by r .

$$\frac{u_{n+1}}{u_n} = r \quad \text{for all } n \geq 1$$

n th Term Formula

If a geometric sequence has first term $a = u_1$ and common ratio r , then:

$$u_n = ar^{n-1}$$

Derivation n th Term of a Geometric Sequence

$$\begin{aligned} u_1 &= a \\ u_2 &= ar \\ u_3 &= ar^2 \\ &\vdots \\ u_n &= ar^{n-1} \end{aligned}$$

Each step multiplies by r , so after $n - 1$ steps we have multiplied by r^{n-1} .

Sum of n Terms

Derivation Sum of Geometric Series

Let $S_n = a + ar + ar^2 + \dots + ar^{n-1}$.

Multiply by r :

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$

Subtract:

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1 - r) &= a(1 - r^n) \end{aligned}$$

Hence, for $r \neq 1$:

$$\boxed{S_n = \frac{a(1 - r^n)}{1 - r}} \quad \text{or} \quad \boxed{S_n = \frac{a(r^n - 1)}{r - 1}}$$

Both forms are equivalent; the second avoids a negative denominator when $r > 1$.

Condition for Convergence and Sum to Infinity

Definition Convergent Series

An infinite series converges if the sum of its terms approaches a finite limit as more and more terms are added. For a geometric series:

$$\sum_{k=1}^{\infty} ar^{k-1} \text{ converges if and only if } |r| < 1.$$

The **sum to infinity** is:

$$S_{\infty} = \frac{a}{1-r} \text{ for } |r| < 1.$$

Warning Divergence

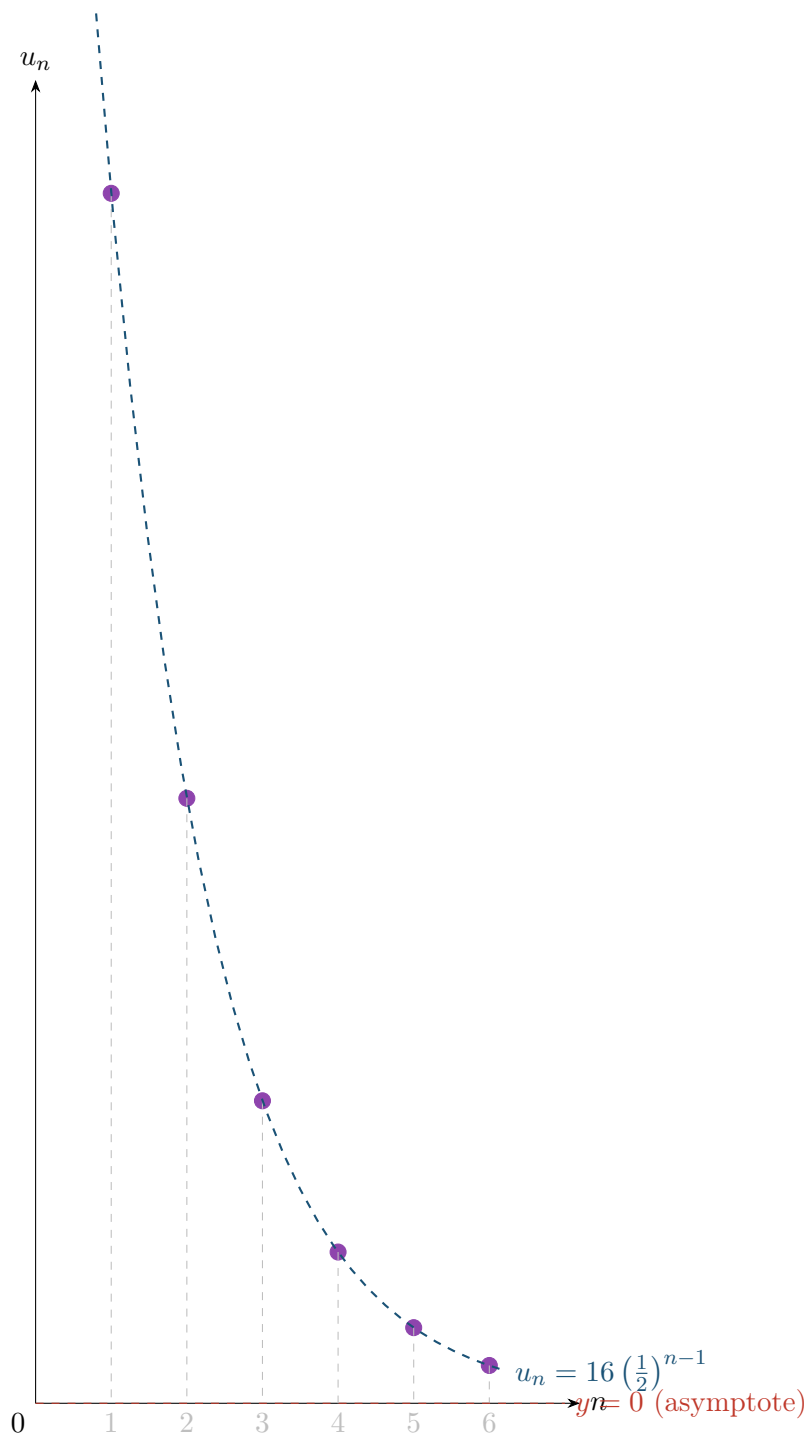
If $|r| \geq 1$, the geometric series does **not** converge (it diverges). Specifically:

- $r > 1$: terms grow without bound \rightarrow sum diverges to infinity.
- $r = 1$: $S_n = na \rightarrow \infty$ (if $a > 0$).
- $r \leq -1$: terms oscillate in magnitude and do not approach a limit.

Do **not** apply $S_{\infty} = a/(1-r)$ unless you have verified $|r| < 1$.

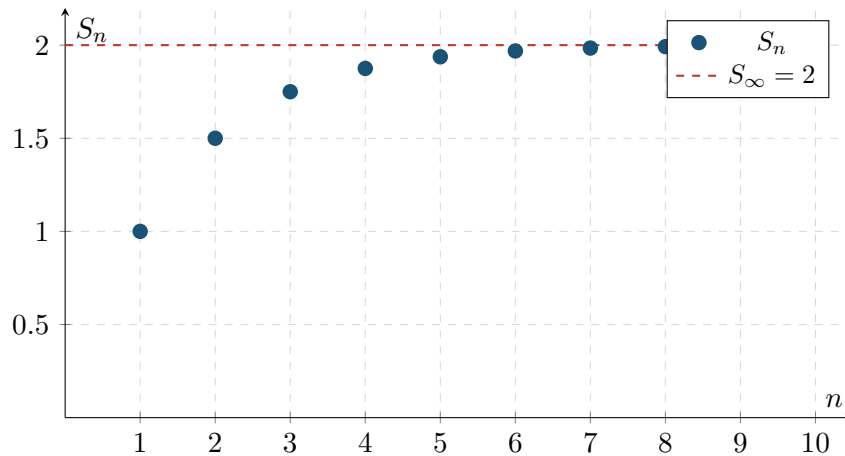
Visualising Geometric Progression

The diagram below shows a geometric progression with $a = 16$, $r = \frac{1}{2}$. The terms decrease exponentially toward zero.



Convergence Diagram: Terms Approaching a Limit

The diagram below illustrates how the terms of a convergent geometric series approach zero as n increases. The **sum to infinity** is the limit of the partial sums S_n as $n \rightarrow \infty$.



Example Geometric Series

The 2nd term of a geometric progression is 6 and the 5th term is 162.

- (a) Find the possible values of the common ratio r and the corresponding first term a .
- (b) For the positive common ratio, find the sum of the first 8 terms.
- (c) Does the sum to infinity exist? If so, find it.

Solution:

$$\begin{aligned} \text{(a)} \quad u_2 &= ar = 6 \\ u_5 &= ar^4 = 162 \end{aligned}$$

Dividing: $\frac{ar^4}{ar} = \frac{162}{6} \implies r^3 = 27 \implies r = 3$ (real). Substituting: $a(3) = 6 \implies a = 2$.

(b) $S_8 = \frac{2(3^8 - 1)}{3 - 1} = \frac{2(6561 - 1)}{2} = 6560$.

(c) Since $r = 3 > 1$, $|r| \not< 1$, so the sum to infinity does **not** exist (the series diverges).

Recurrence Relations

Definition

Definition Recurrence Relation

A **recurrence relation** defines each term of a sequence in terms of one or more preceding terms, together with an initial condition.

$$u_{n+1} = f(u_n), \quad u_1 = \text{given}$$

Types of Recurrence Relations

- **First-order linear:** $u_{n+1} = pu_n + q$ (includes arithmetic and geometric as special cases).
- **Arithmetic-type:** $u_{n+1} = u_n + d$ (equivalent to $u_n = a + (n - 1)d$).
- **Geometric-type:** $u_{n+1} = ru_n$ (equivalent to $u_n = ar^{n-1}$).
- **Non-linear:** e.g. $u_{n+1} = \frac{u_n}{u_n + 1}$, $u_{n+1} = \sqrt{u_n + 2}$.

Using the GC

Exam Tip Generating Terms on GC

On the **TI-84 Plus CE**:

1. Press [mode], set sequential graph.
2. Press [y=], set $n\text{Min} = 1$, $u(n) =$ expression in $u(n - 1)$, $u(n\text{Min}) =$ initial value.
3. Press [2nd] [graph] (TABLE) to view terms.

On the **TI-Nspire CX II**:

1. Add a Lists & Spreadsheet page.
2. In column A, enter $\text{seq}(n, n, 1, 10)$ for indices.
3. In column B, set formula using $\text{when}(n=1, \text{value}, \text{expression})$ or use the recursion template.
4. Alternatively, use the **Graphs** app in sequence mode.

Example Recurrence Relation

A sequence is defined by $u_{n+1} = \frac{2u_n}{u_n + 3}$, $u_1 = 4$.

- (a) Find u_2, u_3, u_4 .
- (b) Guess the limit of the sequence as $n \rightarrow \infty$.

Solution:

$$u_2 = \frac{2(4)}{4 + 3} = \frac{8}{7} \approx 1.1429$$

$$u_3 = \frac{2(8/7)}{8/7 + 3} = \frac{16/7}{29/7} = \frac{16}{29} \approx 0.5517$$

$$u_4 = \frac{2(16/29)}{16/29 + 3} = \frac{32/29}{103/29} = \frac{32}{103} \approx 0.3107$$

The terms appear to be decreasing and approaching 0. For the limit L (if it exists), we set $L = \frac{2L}{L + 3}$, giving $L(L + 3) = 2L$, so $L^2 + 3L - 2L = L^2 + L = L(L + 1) = 0$. Since terms are positive, $L = 0$ is the limit.

Method of Differences

The Core Idea

The **method of differences** is a powerful technique for summing series whose n th term can be expressed as a difference between two consecutive terms of another sequence. It involves **telescoping**: most terms cancel pairwise, leaving only a few terms at the beginning and end.

Definition Telescoping Series

A series $\sum_{k=1}^n (b_{k+1} - b_k)$ telescopes to:

$$\sum_{k=1}^n (b_{k+1} - b_k) = (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{n+1} - b_n) = b_{n+1} - b_1$$

All intermediate terms cancel.

Common Forms for the Method of Differences

- **Partial fractions:** $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.
- $\frac{1}{k(k+2)} = \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+2} \right)$.
- $\frac{1}{(2k-1)(2k+1)} = \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$.
- $\frac{1}{\sqrt{k} + \sqrt{k+1}} = \sqrt{k+1} - \sqrt{k}$ (by rationalising).

Example Method of Differences

Find $\sum_{k=1}^n \frac{1}{k(k+1)}$.

Solution: First, express as partial fractions:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Now write out the sum:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{k(k+1)} &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

As $n \rightarrow \infty$, $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Exam Tip Method of Differences in Exams

When applying the method of differences:

1. Always **write out the first few terms and the last few terms** explicitly to see the cancellation pattern.
2. Verify the cancellation visually — if all intermediate terms cancel, you should have at most $2m$ terms remaining where m is the ‘spread’ of the difference.
3. Check your final expression by testing small values of n (e.g. $n = 1$, $n = 2$) against direct computation.

Sum and Difference of Series

Linearity of Summation

Sigma notation is **linear**:

$$\begin{aligned} \sum_{k=1}^n (a_k + b_k) &= \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \\ \sum_{k=1}^n c a_k &= c \sum_{k=1}^n a_k \quad \text{for any constant } c \end{aligned}$$

Warning Summation of Products

In general, $\sum_{k=1}^n a_k b_k \neq \left(\sum_{k=1}^n a_k\right)\left(\sum_{k=1}^n b_k\right)$. The sum of a product is **not** the product of sums. This is a very common mistake.

Combining Series

When two series are added term-by-term, the sum of the combined series equals the sum of each individual series.

Example Sum of Two Series

Evaluate $\sum_{k=1}^{10} (2k + 3^k)$.

Solution:

$$\begin{aligned} \sum_{k=1}^{10} (2k + 3^k) &= \sum_{k=1}^{10} 2k + \sum_{k=1}^{10} 3^k \\ &= 2 \sum_{k=1}^{10} k + \sum_{k=1}^{10} 3 \cdot 3^{k-1} \\ &= 2 \cdot \frac{10(10+1)}{2} + \frac{3(3^{10}-1)}{3-1} \\ &= 110 + \frac{3(59049-1)}{2} \\ &= 110 + \frac{3 \times 59048}{2} = 110 + 88572 = 88682 \end{aligned}$$

Convergence of Series**Convergent vs. Divergent Series**

A series $\sum_{k=1}^{\infty} u_k$ is said to **converge** if the sequence of partial sums $\{S_n\}$ approaches a finite limit as $n \rightarrow \infty$. Otherwise, the series **diverges**.

Definition Sum to Infinity

If $\lim_{n \rightarrow \infty} S_n = S$ exists as a finite number, then:

$$\sum_{k=1}^{\infty} u_k = S$$

where S is called the **sum to infinity**.

Necessary Condition for Convergence

A **necessary** (but not sufficient) condition for convergence is that the n th term must approach zero:

$$\sum_{k=1}^{\infty} u_k \text{ converges} \implies \lim_{n \rightarrow \infty} u_n = 0$$

Warning The Converse is False

If $\lim_{n \rightarrow \infty} u_n = 0$, the series may still diverge. The classic counter-example is the **harmonic series**:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \text{diverges, even though } \frac{1}{k} \rightarrow 0.$$

However, for H2 Math, you will primarily work with geometric series, where the condition $|r| < 1$ is both necessary and sufficient for convergence.

Convergence of Geometric Series

For the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$:

$$|r| < 1 \implies \text{converges to } S_{\infty} = \frac{a}{1-r}$$

$$|r| \geq 1 \implies \text{diverges}$$

Exam Tip Checking Convergence

When a question asks ‘find the sum to infinity of a geometric series’, always:

1. Identify a and r .
2. Check $|r| < 1$ explicitly in your working.
3. If $|r| < 1$, state $S_{\infty} = \frac{a}{1-r}$.
4. If $|r| \geq 1$, write ‘the sum to infinity does not exist (the series diverges)’.

Common Misconceptions**Arithmetic vs. Geometric Confusion**

- Arithmetic: **adding** a constant difference d between terms.
- Geometric: **multiplying** by a constant ratio r between terms.
- Check: Is the difference between consecutive terms constant? If yes, it is arithmetic. Is the ratio constant? If yes, it is geometric.

 u_n vs. S_n

- S_n is the **sum** of n terms, not the n th term.
- $u_n = S_n - S_{n-1}$ only works for $n \geq 2$.
- Do not confuse S_n (sum) with u_n (term).

Sigma Notation Index Errors

- $\sum_{k=1}^n u_k$ starts at $k = 1$, not $k = 0$, unless specified.
- $\sum_{k=1}^n ar^{k-1}$ has n terms; $\sum_{k=0}^{n-1} ar^k$ is equivalent.
- Be careful when shifting indices — check that the number of terms and the first/last terms match.

Common Ratio of Geometric Series

- r can be **negative** — this causes alternating signs in the terms.
- $|r| < 1$ is the condition for convergence, **not** $r < 1$.
- A negative r with $|r| < 1$ (e.g. $r = -0.5$) still converges.

Telescoping Series Pitfalls

- Always factor out constants correctly when using partial fractions.
- Write out at least 3 terms at the start and 3 terms at the end to see the cancellation pattern.
- Remember that $\sum_{k=1}^n (b_{k+1} - b_k) = b_{n+1} - b_1$, **not** $b_n - b_1$.

Infinite Series Sum

- You cannot compute S_∞ for a divergent series.
- For non-geometric series, convergence is not always obvious.
- In H2 Math, the method of differences often gives sums that converge (e.g. $\sum \frac{1}{k(k+1)} \rightarrow 1$).

Connections to Other Topics

Sequences and series are fundamental to many other areas of H2 Mathematics:

- **Mathematical Induction (Topic 2.2):** Induction is frequently used to prove formulas for S_n , especially for arithmetic and geometric series, and for sums found via the method of differences. For example, proving $\sum_{k=1}^n \frac{1}{k(k+1)} = \frac{n}{n+1}$ by induction is a standard exercise.
- **Calculus — Integration:** The sum to infinity of a geometric series is related to the area under a curve through Riemann sums. The integral $\int_0^\infty e^{-x} dx = 1$ parallels the geometric series $\sum_{k=1}^\infty e^{-(k-1)} = \frac{1}{1-e^{-1}}$.
- **Binomial Theorem:** The binomial expansion $(1+x)^n$ is a finite series for integer n , but becomes an infinite series (convergent for $|x| < 1$) for non-integer n .
- **Maclaurin Series:** The Maclaurin series for functions like e^x , $\sin x$, and $\cos x$ are infinite series that converge to the function value. Understanding convergence of geometric series provides intuition for the radius of convergence of Maclaurin series.
- **Complex Numbers:** Geometric series appear when summing powers of complex numbers, particularly in De Moivre's theorem applications and roots of unity.
- **Statistics and Probability:** The geometric distribution (probability of first success on the n th trial) leads to series sums for expected values and variances.
- **AP/GP in Compound Interest and Annuity Problems:** Arithmetic and geometric progressions model real-world financial mathematics, including loan repayments, savings plans, and investment growth.

Exam Tip Exam Strategy

In H2 Math Paper 1, sequences and series questions often appear as:

- A stand-alone question on arithmetic or geometric series (5–7 marks).
- A method of differences question combined with partial fractions (4–6 marks).
- Part of a multi-part question linking to mathematical induction, binomial theorem, or Maclaurin series.

Always write down the relevant formulas first — partial marks are awarded for formula recall even if your subsequent algebra is incorrect.

Quick Reference — Key Formulas

Concept	Formula
Arithmetic n th term	$u_n = a + (n - 1)d$
Arithmetic sum	$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + \ell)$
Geometric n th term	$u_n = ar^{n-1}$
Geometric sum (n terms)	$S_n = \frac{a(1 - r^n)}{1 - r} \quad (r \neq 1)$
Sum to infinity	$S_\infty = \frac{a}{1 - r} \quad (\text{only if } r < 1)$
Convergence condition	$ r < 1$
n th term from sum	$u_n = S_n - S_{n-1} \quad (n \geq 2)$
Telescoping sum	$\sum_{k=1}^n (b_{k+1} - b_k) = b_{n+1} - b_1$